J. Fluid Mech. (1982), vol. 114, pp. 251–260
Printed in Great Britain



# The onset of convection in fluid layers heated rapidly in a time-dependent manner

By BHARAT S. JHAVERI† AND G. M. HOMSY
Department of Chemical Engineering, Stanford University, California 94305

(Received 13 November 1980 and in revised form 5 June 1981)

We treat the onset of convective motions for the case in which the base-state density profile is evolving in time. The formulation is in terms of random forcing which we take to be thermodynamic in origin, following our earlier work (see Jhaveri & Homsy 1980). Experimental evidence is reviewed which clearly demonstrates the need for such a stochastic formulation. The randomly forced initial-value problem is solved numerically at high Rayleigh numbers in the mean-field approximation for both a step change and linear temporal increase in surface temperature. The numerical results give both an expected value for the onset time for which convection is measurable and the variance of that expected value. The results are in good agreement with available experiments.

## 1. Introduction

In this work we address the problem of the determination of the onset time of convection in initially quiescent fluid layers whose base temperature profile is developing with time. There has been little analytical progress on this class of problems since it was critically reviewed by Homsy (1973). The earliest approach to the analytical determination of onset times consisted of 'freezing' the diffusing base state at a given time and determining its marginal stability; the time thus appears only parametrically (Lick 1965; Currie 1967). This approach proceeds from the assumption that disturbances are growing faster than the base state is evolving. An alternative initial value approach has been taken by Foster (1965, 1968) and afterwards by Mahler, Schechter & Wissler (1968), Mahler & Schechter (1970) and Gresho & Sani (1971), amongst others. The linear disturbance equations are Fourier decomposed and the spatial dependence is removed by taking appropriate inner products. Evolution equations of the form

$$\frac{dx_i(t)}{dt} = a_{ij}(t)\, x_j, \tag{1.1}$$

are derived for the time-dependent Fourier amplitudes, $x_i(t)$, with the wavenumber and Rayleigh number appearing as parameters in equation (1.1). The strength of convective motion is defined in terms of a suitable norm $\|x_i(t)\|$ for the growing disturbance amplitudes. The amplitude equations are then integrated numerically subject to some initial conditions and the growth of disturbances is followed as a function of time. Parametric dependence of the evolution (1.1) on the wavenumber is removed by choosing the particular wavenumber for which the evolution is fastest. Convection

† Current address: Department of Physics, University of Utah, Salt Lake City, Utah.

patterns having this wavenumber are supposed to occur when the initial convective strength of the disturbances, $\|x_i(0)\|$, has amplified by a suitable factor, $A$. That is, at the onset of convection the equality

$$\|x_i(t_0)\| = A\|x_i(0)\| \tag{1.2}$$

holds. The amplification factor is usually found by matching the predicted evolution with experiments. Drawbacks with this approach are discussed below. Finally, a somewhat different approach to this problem has been taken by Homsy (1973). Using the method of energy, a lower bound to onset times may be determined. Any convective motion is shown to decay exponentially for times below this lower bound. These bounds are, in general, quite conservative (see, for example, Gumerman & Homsy 1975; Wankat & Homsy 1977).

These theories are unable to explain many of the observations concerning the measurement of the onset time. Many investigations concerned with the experimental determination of the onset time and the form of motion at onset have been reported in the literature; for reviews see Davenport (1972), and Spangenberg & Rowland (1961). We wish to discuss in detail the experiments of Blair & Quinn (1969), because of their accurate flow visualization and measurements of onset times. Onset experiments were carried out in a cylindrical apparatus of large aspect ratio in order to approximate the condition of infinite horizontal extent. Gas adsorption of sulphur dioxide into water was used to create the buoyancy force. Since the solution of sulphur dioxide is denser than water, transient experiments were carried out by subjecting an initially quiescent fluid layer of uniform composition to a step change in the concentration of sulphur dioxide at the upper boundary. The resulting concentration profiles were visualized using schlieren photography. As the gas was diffusing by a molecular mechanism, a layer of concentrated sulphur dioxide developed below the upper boundary. Initially the bottom surface of this diffusion layer was smooth and steady as it grew. Then the concentrated solution collected along circular rings at the edge of the diffusion layer. Rings then detached from the upper surface and fell into the lighter liquid below. Onset times were measured both visually and by monitoring the rate of uptake of gas at the upper boundary. Since convection augments transport into the liquid, the onset of convection is accompanied by a corresponding increase in the rate of adsorption over that due to diffusion alone. Measured onset times thus depended upon the smallest increment in Sherwood number (dimensionless surface flux) above the diffusive value the measuring instrument could detect.

The following general observations were made. Rings originated at random locations below the surface of the developing diffusion layer. The appearance and detachment of these rings continued at irregular intervals. For example in one experiment in which an onset time of 27 s was measured, the times of the origination and detachment of rings varied from 24 to 30 s. Also the strength of convective motion in different rings was different at a given time. When the same experimental run was carried out in two different apparatuses, no substantial difference in the measurements of onset time was found. Thus it was concluded that initial perturbations had no effect on the onset time. The wavenumber at the onset was measured by counting the number of rings per unit area. The wavenumber was found to increase with the Rayleigh number. For highly supercritical Rayleigh numbers the onset of convection was observed long before the bottom of the fluid experienced any significant increase in the sulphur dioxide concentration. The onset time and the wavenumber at onset then should be independent of the depth of the layer. Dimensional considerations give these asymptotic limits as:

$$Rt_0^{\frac{3}{2}} = R_t = \text{constant}, \quad R\alpha_0^{-3} = \alpha_t = \text{constant}, \tag{1.3}$$

Here $R$ is the Rayleigh number based on the total depth, and $t_0$ and $\alpha_0$ are respectively the dimensionless time and the wavenumber at onset. These two asymptotic relations were also verified experimentally.

Davenport & King (1974) carried out transient experiments in which the motion was driven by temperature gradients. Their apparatus was isolated from mechanical vibrations through use of a vibration-free table. Some of their experiments were done with a surface temperature which increased linearly with time. Their experimental results showed that the condition of the onset of observable convection at a gas–liquid interface and a solid–liquid interface were the same if surface waves, meniscus and surface tension gradient effects were eliminated from the experiment. For this manner of linear heating, dimensional considerations give the asymptotic limits as:

$$Rt_0^{\frac{5}{2}} = R_t, \quad R\alpha_0^{-5} = \alpha_t. \tag{1.4}$$

The asymptotic onset times, given by $R_t(Pr)$, were measured for a wide range of Prandtl numbers.

Consider the evolution equation (1.1). The initial value approach assumes that all evolutions (rings in experiments of Blair & Quinn) have identical initial conditions. Thus it predicts that all rings will evolve identically and become observable at the same time, a prediction which is contrary to the experimental observations. Blair & Quinn (1969), and Davenport & King (1973) attempted to compare their observations with predictions of the initial value approach and no unique value of the amplification factor was found. In fact values of $A$ ranging between $10^3$ and $10^8$ were necessary to fit the experiment with the amplification theory. However, the wavenumber at the onset of convection was found to be 'the fastest growing', as predicted by the initial-value approach. This is intuitive if one assumes that fluctuations of all wavenumbers are present in the fluid; the fastest growing one will be detected first. Thus, subjective decisions regarding the choice of initial conditions, definition of strength of convection in terms of a norm, and the choice of amplification factor, have to be made by the analyst. Another drawback lies in the assumption that the disturbances are present only at an initial instant. Disturbances are in general present throughout the evolution and this may substantially affect the evolution. Finally, much of the previous work neglects the nonlinear effects. When the convective motion becomes observable, disturbances may no longer be considered infinitesimal and nonlinear effects may not be neglected. Indeed, the use of measurements of *mean* transport in order to define onset of convection implies finite-amplitude motions.

In light of these observations, it seems appropriate to develop a model in which disturbances are treated statistically. Many of the features of the formulation of a statistically based theory have been given by us in a previous paper (Jhaveri & Homsy 1980; hereinafter referred to as I). In the present paper, we extend this to highly supercritical conditions in order to compare the predictions of the stochastic theory with experiment.

## 2. Stochastic formulation

It is obvious from previous considerations that more satisfactory modelling is required to explain the experimental observations. We arrive at the stochastic formulation by reconsidering these observations. Since the measured onset times correspond to the average convective transport by all evolutions and since the onset times were not significantly different in two independent experimental runs in different apparatuses, an immediate conclusion is that all evolutions in an experiment were statistically correlated so as to have reproducible average transport. Hence they must have evolved from a statistical distribution of initial values. It is unlikely to have an almost identical distribution of initial values unless caused by some random forcing common to the experiments. Such random forcing could be caused, for example, by building vibrations. It is then natural to introduce random forcing in the governing disturbance equations. As in I, in order to provide an upper bound to the onset times in *any* physical system, and a sufficient condition for instability, we introduce this forcing as thermodynamic fluctuations, which are always present in any macroscopic system. Our results will then hold for all experiments devoid of perturbations of mechanical nature. The advantages of introducing random forcing are twofold. First, in solving for the state of rest, random forcing immediately specifies the statistics of initial values. Second, it takes into account the continuous presence of noise during evolution.

As a further step to improve modelling, we consider the possibility of nonlinear interactions to the lowest order modification of the horizontal mean temperature and hence the Nusselt number. Since the disturbances just become observable at the onset of convection, it is appropriate to neglect higher-order interactions at this time. We discuss these approximations in detail below. These considerations turn the deterministic evolution equation (1.1) into a random one, *viz.*

$$\frac{dx_i(t)}{dt} = a_{ij}(t)\,x_j + Q_{ijk}\,x_j\,x_k + \epsilon f_i(t), \tag{2.1}$$

where $f_i(t)$ is a random variable, $Q_{ijk}$ is a nonlinear coupling tensor, and $\epsilon$ is the strength of the forcing. As discussed in I, since the statistics of the forcing are available from statistical thermodynamics and the statistics of initial value $x_i(0)$, are easily calculated in terms of the statistics of forcing, the formulation of a statistical initial-value problem is complete for the random evolution $x_i(t)$.

Now we are ready to outline broadly the solution process used to evaluate onset times from the random evolution $x_i(t)$. It is important to first give a consistent definition of onset time. The horizontal extent of fluid in an experiment is considered large enough to generate a large number of evolutions approximating the ensemble for the random evolution $x_i(t)$. With this assumption, if an experiment is repeated in another apparatus with identical forcing, the statistical mean value properties of the realized evolutions are reproducible. Let $M_s$ correspond to the smallest value of mean Nusselt number $\langle Nu\rangle$ above unity the measuring instrument can detect. We define the onset of convection when the mean Nusselt number $\langle Nu\rangle$ first reaches the value $M_S$. From the solution process for (2.1), we first arrive at the curve for $\langle Nu(t)\rangle$. Then the onset of convection is defined through

$$\langle Nu(t_0)\rangle = M_S. \tag{2.2}$$

The onset time $t_0$ is thus readily calculated from the theoretical evolution of $\langle Nu(t)\rangle$. The wave number at the onset of convection is still considered to be the fastest growing one.

This completes the stochastic formulation and the outline of the solution process in general terms. As mentioned earlier we have analysed a weakly supercritical convection problem in I with this formulation. We extend the analysis of I to highly supercritical conditions under which the experiments were performed.

We present below a fairly simple theoretical model of onset experiments. The physical system consists of a fluid layer confined between two infinitely extended horizontal planes. The fluid is considered to be locally incompressible in the Boussinesq approximation. The bounding surfaces are considered isothermal, impermeable and stress-free. The fluid layer is initially isothermal and quiescent. At the instant $t = 0$, the bottom surface is heated in a time-dependent manner. We consider two distinct cases of heating; case $A$ in which the fluid is subjected to a sudden increase in the bottom temperature, and case $B$ in which the bottom temperature is increased linearly in time. We start with the dimensionless governing equations for the disturbances with inclusion of thermodynamic fluctuations as random forces which take the same form as equation (2.1) in I except for the term proportional to the Rayleigh number, since the base temperature profile is now time dependent. These equations are:

$$\left.\begin{aligned} \frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j} &= -\frac{\partial p}{\partial x_i}+Pr\,T\,\delta_{i3}+Pr\frac{\partial^2 u_i}{\partial x_j\partial x_j}+\frac{\partial S_{ij}}{\partial x_j},\\ \frac{\partial T}{\partial t}+u_j\frac{\partial T}{\partial x_j} &= R(r(z,t))\,u_i\delta_{i3}+\frac{\partial^2 T}{\partial x_j\partial x_j} = 0,\\ \frac{\partial u_j}{\partial x_j} &= 0. \end{aligned}\right\} \tag{2.3}$$

Here $R$ is the Rayleigh number, $S_{ij}$ is the fluctuating stress tensor, and $r(z,t)$ is the gradient of the diffusing base temperature profile. For case $A$ of sudden heating,

$$r(z,t) = 1+2\sum_{n=1}^{\infty}\cos(n\pi z)\exp[-n^2\pi^2 t]. \tag{2.4}$$

For case $B$ of linear heating,

$$r(z,t) = t+(2/\pi^2)\sum_{n=1}^{\infty}\cos(n\pi z)\,(1-\exp[-n^2\pi^2 t])/n^2. \tag{2.5}$$

The boundary conditions are $u_i = T = 0$, $z = 0, 1$.

In equation (2.3), random terms are included only in the momentum equation for reasons discussed in I. The temperature difference $\Delta T$ in the definition of the Rayleigh number $R$ is taken as actual temperature difference across the layer in case $A$ and $(\gamma l^2/\kappa)$ in case $B$. Here $\gamma$ is the rate of increase in bottom temperature, $l$ is the length of the layer and $\kappa$ is the coefficient of thermal diffusion.

Certain assumptions have to be made and justified before we proceed with the solution process. It is generally observed that in the transient experiments of this kind, the first convective motion appears to have a two-dimensional form. Rings in the transient experiments of Blair & Quinn were axisymmetric at the onset of convection. In the transient experiments of Spangenberg & Rowland (1961) in a rectangular geometry, the first motion occurred in the form of two-dimensional plunging sheets.

It is justifiable to restrict the analysis to two dimensions, at least until the first motion appears, and we do so here. The second assumption concerns the Fourier decomposition of the disturbance equations. Since the base temperature profile is developing by thermal diffusion in the vertical direction, we pay careful attention to the vertical structure of convection and include as many normal modes into the decomposition as required for convergence. However we simplify the horizontal dependence and retain only one mode in the horizontal structure. Such a truncation is justified since in the experiments of Blair & Quinn (1969) and Spangenberg & Rowland (1961), large-scale features of convective motion at the onset were dominated by one horizontal mode. Thus it is appropriate to represent the horizontal structure by a single mode, at least for a short finite time after the onset of motion. The amplitude equations obtained after such decomposition are also called the single-$\alpha$ mean-field equations. The mean-field equations contain an $O(1)$ constant distinguishing different horizontal planforms like rolls, rings, hexagons, etc. Use of the mean field approximation allows comparison between the analysis with two-dimensional rolls and experiments with rings. Mean field equations of this type have been successfully used in the past to predict transport and explain some of the large-scale features of natural convection (see Gough, Spiegel & Toomre 1975). With such a truncation the only effect of nonlinear interactions is in the modification of the horizontal mean temperature. This is equivalent to restricting the first index $a$, in the modal expansions (2.4, 2.5) of I to values of 0 and 1. Finally, the dimensionless number $\theta$ giving the variance of the random stress is assumed to be constant in time although the average temperature of the fluid layer is time dependent. This is justified since $\theta$ is of thermodynamic order and the variation in the absolute temperature of the layer will not drastically change its order.

## 3. Solution

With the assumptions made above, the Fourier decomposition of dependent variables in the horizontal direction can be written as

$$w = c(z,t)\cos(\alpha x), \quad T = d_0(z,t) + d(z,t)\cos(\alpha x), \tag{3.1}$$

with associated expansions for $u$ and $p$. Substituting these expansions in the dimensionless governing equations and forming the appropriate inner products to remove the horizontal space dependence, we obtain in the usual way, the following partial differential equations for the Fourier amplitudes $c(z,t)$, $d(z,t)$ and $d_0(z,t)$:

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{D^2-\alpha^2}{\alpha^2}\right)c(z,t) &= Pr\left(\frac{D^2-\alpha^2}{\alpha}\right)^2 c(z,t) - Pr\, d(z,t) + f(z,t),\\
\frac{\partial}{\partial t}d(z,t) &= Rr(z,t)\,c(z,t) + (D^2-\alpha^2)\,d(z,t) - cD(d_0(z,t)),\\
\frac{\partial}{\partial t}d_0(z,t) &= D^2 d_0(z,t) - \tfrac{1}{2}D(c(z,t)\,d(z,t));
\end{aligned} \tag{3.2}$$

where $D = \partial/\partial z$, and

$$f(z,t) = \frac{\alpha}{\pi}\int_0^{2\pi/\alpha}\left\{\frac{\partial^2 S_{xx}}{\partial x\,\partial z} + \frac{\partial^2 S_{xz}}{\partial z^2}\right\}\left(-\frac{1}{\alpha}\sin(\alpha x)\right) - \left(\frac{\partial S_{zx}}{\partial x} + \frac{\partial S_{zz}}{\partial z}\right)\cos(\alpha x)\Big\}\,dx. \tag{3.3}$$

If we expand the amplitudes into normal modes in $z$ and proceed as in I, nonlinear terms will become convolution sums; refer to equations (2.6)–(2.8) of I. It turns out that it is economical and convenient to use spectral methods to evaluate these nonlinear terms (Orszag 1971). Let the product terms in equation (3.2) be represented symbolically as

$$\left.\begin{aligned} g(z,t) &= r(z,t)\,c(z,t), \\ s(z,t) &= c(z,t)\,Dd_0(z,t), \\ h(z,t) &= \tfrac{1}{2}D[c(z,t)\,d(z,t)]. \end{aligned}\right\} \qquad (3.4)$$

Then equations (3.2) may be formally Fourier-transformed in the $z$ direction to yield

$$\left.\begin{aligned} \frac{dc_k(t)}{dt} &= -Pr(\alpha^2+k^2\pi^2)\,c_k(t)+\frac{\alpha^2}{\alpha^2+k^2\pi^2}\{Prd_k(t)-f_k(t)\}, \\ \frac{d}{dt}d_k &= Rg_k(t)-(\alpha^2+k^2\pi^2)\,d_k(t)+S_k(t), \\ \frac{d}{dt}(d_{0k}) &= -\pi^2k^2d_{0k}(t)+h_k(t). \end{aligned}\right\} \qquad (3.5)$$

Here the subscript $k$ refers to the $k$th Fourier component. The system (3.5) is subject to the initial conditions

$$d_k(0) = d_{0k}(0) = 0; \qquad (3.6a)$$

$c_k(0)$ is related to the statistics of $f$, as shown in I, and thus $c_k(0)$ is a Gaussian variable with zero mean and correlation,

$$\langle|c_k(0)\,c_l(0)|\rangle = \left(\frac{n}{2}\right)^2\Theta\left(\frac{2\alpha}{\pi}\right)\left(\frac{1}{Pr}\right)\left(\frac{\alpha^2}{\alpha^2+k^2\pi^2}\right)\delta_{kl}. \qquad (3.6b)$$

The forcing $f_k(t)$ is also Gaussian with zero mean and correlation:

$$\langle|f_k(t_1)|\cdot|f_l(t_2)|\rangle = \left(\frac{n}{2}\right)^2\left(\frac{4\alpha^3\Theta}{\pi}\right)\delta_{kl}\,\delta(t_1-t_2). \qquad (3.7)$$

In these equations, $n$ is the number of discrete points in the finite Fourier transform pair; see discussion below on the method of solution, and Jhaveri (1979).

We have solved the problem numerically by Monte Carlo methods as described in I; however here we have the problems of (i) evaluation of the nonlinear terms $s(z,t)$ and $h(z,t)$, cf. equation (3.4); and (ii) carrying many modes in the $z$ direction to resolve the spatial structure of the evolving base state, $r(z,t)$ properly, cf. equation (3.4).

We have used spectral methods to solve the partial differential equations given by (3.2). The implementation of the spectral method, in which the nonlinear terms in equation (3.4) are evaluated in real space on a discrete grid, and the time-stepping is done in Fourier-space by integrating equation (3.5) is discussed in detail in Jhaveri (1979). The main steps for the numerical simulation are as follows:

Step 1. Deterministically integrate the system of equations (3.5) from deterministic initial conditions to determine both the number of vertical modes required for convergence and the fastest growing wavenumber $\alpha_0$.

Step 2. Generate $N$ samples for each $c_k$, $|k| \leqslant n$ distributed with zero mean and variance given by (3.6$b$). The $N$ samples for each $d_k$ and $d_{0k}$ are initially zero.

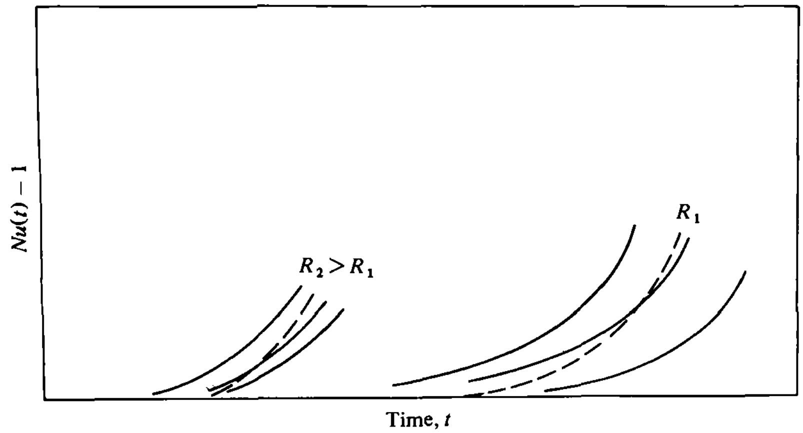


FIGURE 1. Schematic diagram of typical numerical results showing convective transport for individual realizations, and the mean (dashed line), as a function of increasing $R$.

Step 3. Evaluate the spectrum of nonlinear product terms, $g_k$, $s_k$ and $h_k$, by the spectral method; Orszag (1971) and Jhaveri (1979).

Step 4. Deterministically integrate (3.5) for a step size $h$.

Step 5. Generate $N$ samples for random increment $b_k(t_1, t_2)$ for each $c_k$, distributed with zero mean and variance related through forcing as,

$$\langle b_k(t_1, t_2)^2 \rangle = \left(\frac{n}{2}\right)^2 \left(\frac{4\alpha^3 \Theta}{\pi}\right) h.$$

Step 6. Evaluate the Nusselt number for such samples and then calculate the mean Nusselt number $\langle Nu(t) \rangle$ from this approximate ensemble.

Step 7. If $\langle Nu(t) \rangle \geqslant M_s$, *stop*. Otherwise repeat from step 3.

As mentioned earlier, since higher-order nonlinear interactions become significant at long times, the assumptions involved in deriving the mean-field equations become invalid. Thus, we carry out the simulation only until significant convective motion commences.

## 4. Results and conclusions

The calculations were carried out for $Pr = 7$ and $\Theta = 3{\cdot}43 \times 10^{-7}$ for both cases of heating. The value of $\Theta$ is appropriate for most liquids at room temperature; see I.

The results of these calculations were of course similar to those reported in I. Figure 1 in which we plot the convective transport *vs.* time, gives a schematic diagram of the results. For any given $R$, the realizations are distributed about some mean, see, for example, figure 2 of I. We find that for increasing $R$, measurable convection begins sooner, as is to be expected, and the variance of realizations about the mean is reduced.

For case $A$ of sudden heating, the Rayleigh number was increased until the asymptotic limits were obtained, as given by equations (1.3). Onset times corresponding to three different sensitivities of the measuring instrument were evaluated. The results are presented in table 1. We see that in the limit of large Rayleigh numbers ($R \geqslant 30R^*$), the asymptotic relation for onset time given by equation (1.3) holds approximately.

| Rayleigh number $R^* = (27/4)\pi^4$ | $R_t = Rt_0^{\frac{3}{2}}$ $M_s = 1{\cdot}01$ | $M_s = 1{\cdot}05$ | $M_s = 1{\cdot}1$ | $R\alpha_0^{-3}$ |
|---|---|---|---|---|
| $1{\cdot}5\,R^*$ | 1816 | — | — | 68 |
| $10\,R^*$ | 346 | 382 | 398 | 146 |
| $30\,R^*$ | 319 | 350 | 364 | 225 |
| $100\,R^*$ | 318 | 348 | 362 | 246 |
| $300\,R^*$ | 314 | 343 | 357 | 264 |
| $1000\,R^*$ | 321 | 349 | 363 | 264 |

TABLE 1. Variation of onset time and the wavenumber at onset with the Rayleigh number.

That the asymptotic results are only approximately constant is attributed to the sampling error involved in the Monte Carlo simulation. It should also be noted that with the decrease in sensitivity (higher value of $M_s$), the corresponding increase in onset time is not linear. This is in agreement with one's intuition that different measuring instruments should not yield widely different onset times.

Our Monte Carlo calculations also allow us to predict the variance of onset times. If each evolution is defined to become observable when its Nusselt number exceeds the value of $1{\cdot}01$, and if $\Delta t$ denotes the time interval during which all evolutions become observable, then in the asymptotic limit, we find

$$R(\Delta t)^{\frac{3}{2}} \simeq 36. \tag{4.1}$$

In the experiments of Blair & Quinn, the asymptotic value measured was

$$Rt_0^{\frac{3}{2}} \simeq 300$$

which compares favourably with $R_t \simeq 350$ from table 1.

In a typical experiment described by Blair & Quinn, when the onset time of 27 s was measured, the time interval of ring origination was from 24 to 30 s. For their data this spread corresponds to

$$R(\Delta t)^{\frac{3}{2}} = 31,$$

again in good agreement with our numerical results, for example, equation (4.1). The experiments were carried out for large Schmidt (Prandtl) numbers, so the corresponding onset times for $Pr = 7$ should be somewhat higher. With this consideration, our asymptotic results for onset time and the time interval of ring origination are in excellent agreement with the experimental measurements.

For case $B$ of linear heating, the asymptotic results were calculated using $R = 10^8$ for which the fastest growing wave number corresponds to $\alpha_0 = 11$, (Foster 1965). The asymptotic results in the form of equation (1.4) are presented in table 2 for three different sensitivities. Our analysis corresponds to Davenport & King's (1974) experiments where a linear surface temperature decay was generated using a thermoelectric cooler. For $Pr = 7$, the measured experimental $R_t$ found by extrapolation from their figure 3, is

$$R_t = 973.$$

Again, excellent agreement is found between the analysis and the experiments.

| $R$ | $R_t = Rt_0^{\frac{3}{2}}$ | | |
|---|---|---|---|
| | $M_s = 1{\cdot}01$ | $M_s = 1{\cdot}05$ | $M_s = 1{\cdot}1$ |
| $10^8$ | 911 | 995 | 1035 |

TABLE 2. Asymptotic results for case $B$ of linear heating.

We have calculated onset times of convection using a stochastic theory which are in good agreement with available experiments. It is necessary to estimate the strength of the random forcing (the parameter $\Theta$ above). In the case of the experiments of Blair & Quinn, the appropriate value would be that given by a statistical theory of concentration fluctuations, since theirs was a mass transport experiment. The order of magnitude of $\Theta$ however would be the same. In the case of the experiments of Davenport & King, the present estimate of $\Theta$ is appropriate. We feel it is remarkable that our theory, in which the strength of the forcing is estimated from statistical thermodynamics, is in such good agreement with experiment. While this does not prove that these instabilities were due directly to thermodynamic fluctuations, it does indicate that the forcing present in these experiments was of the same order of magnitude.

We wish to acknowledge the support of this work by the U.S. Department of Energy, Office of Basic Energy Sciences.

## REFERENCES

BLAIR, L. M. & QUINN, J. A. 1969 *J. Fluid Mech.* **36**, 385.

CURRIE, I. G. 1967 *J. Fluid Mech.* **29**, 337.

DAVENPORT, I. F. 1972 *The initiation of natural convection by time-dependent profiles.* Ph.D. thesis, University of California, Berkeley, California.

DAVENPORT, I. F. & KING, J. C. 1974 *Int. J. Heat Mass Transfer* **17**, 77.

FOSTER, T. D. 1965 *Phys. Fluids* **8**, 1249.

FOSTER, T. D. 1968 *Phys. Fluids* **11**, 1257.

GOUGH, D. O., SPIEGEL, E. A. & TOOMRE, J. 1975 *J. Fluid Mech.* **68**, 695.

GRESHO, P. M. & SANI, R. L. 1971 *Int. J. Heat Mass Transfer* **14**, 207.

GUMERMAN, R. J. & HOMSY, G. M. 1975 *J. Fluid Mech.* **68**, 191.

HOMSY, G. M. 1973 *J. Fluid Mech.* **60**, 129.

JHAVERI, B. S. 1979 Rayleigh–Bénard Convection from Thermodynamic Fluctuations, Ph.D. thesis, Stanford University, Stanford, California.

JHAVERI, B. & HOMSY, G. M. 1980 *J. Fluid Mech.* **98**, 329.

LICK, W. 1965 *J. Fluid Mech.* **21**, 565.

MAHLER, E. G., SCHECHTER, R. S. & WISSLER, E. H. 1968 *Phys. Fluids* **11**, 1901.

MAHLER, E. G. & SCHECHTER, R. S. 1970 *Chem. Engng Sci.* **25**, 955.

ORSZAG, S. A. 1971 *Stud. Appl. Math.* **4**, 293.

SPANGENBERG, W. G. & ROWLAND, W. R. 1961 *Phys. Fluids* **4**, 743.

WANKAT, P. C. & HOMSY, G. M. 1977 *Phys. Fluids* **20**, 1200.